INVERTIBLE KNOT CONCORDANCES AND PRIME KNOTS

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1. Introduction

Kirby and Lickorish [1] showed that every knot in S^3 is concordant to a prime knot, equivalently, every concordance class contains a prime knot. Generalizations appear in [3, 4, 5, 9]. Sumners [11] introduced the notion of invertible concordance. We prove here that the Kirby and Lickorish's result can be strengthened:

Theorem 1.1. *Every knot in S^3 is invertibly concordant to a prime knot.*

Corresponding to invertible concordance there is a group, the *double concordance group*, studied in [2, 6, 10]. A consequence of our work is that every double concordance class contains a prime knot.

2. Definitions and basic results

In all that follows manifolds and maps will be smooth and orientable. Let I denote the interval $[0, 1]$.

A *link* of n components, L, is a smooth pair (S^3, l) where l is a smooth oriented submanifold of S^3 diffeomorphic to n disjoint copies of S^1. A *knot* K is a link of one component. Two links, L_1 and L_2, each of n components, are called *concordant* if there exists a proper smooth oriented submanifold w of $S^3 \times I$, with $\partial w = (l_1 \times 0 \cup (-l_2) \times 1)$ and w diffeomorphic to n disjoint copies of $S^1 \times I$. Let $(W; L_1, L_2)$ denote $(S^3 \times I, w)$ the concordance between L_1 and L_2. If $(W_1; L_1, L_2)$ and $(W_2; L_2, L_3)$ are two concordances with a common boundary component (oriented oppositely) we can then paste W_2 to W_1 along L_2 to get $(W_1 \cup W_2; L_1, L_3)$.

A concordance $(W; L_1, L_2)$ is said to be *invertible at* L_2 if there is a concordance $(W'; L_2, L_1)$ such that $(W \cup W'; L_1, L_1)$ is diffeomorphic to $(L_1 \times I; L_1, L_1)$, the product concordance of L_1. Given the above situation, we say that L_1 *is invertibly concordant to* L_2, and L_2 *splits* $L_1 \times I$. In the same manner, concordance and invertible concordance can be defined for knots and links in the solid torus $S^1 \times D^2$.

A submanifold N with boundary is said to be *proper* in a manifold M if $\partial N = N \cap \partial M$. Let B^3 denote the standard closed 3-ball $\{x \in \mathbb{R}^3 \mid |x| \leq 1\}$. An *$n$-tangle* T is a smooth pair (B^3, λ) where λ is a proper embedding of n disjoint copies of the interval I into B^3. Throughout this paper, an embedding means either the map or the image. Let U_n denote a trivial n-tangle, *i.e.*, U_n consists of n unlinked unknotted arcs. For example, U_1 is the unknotted standard ball pair (B^3, I). For $n = 2$, see Figure 1.

Concordances and invertible concordances between tangles can be defined in a similar way as for links. However, the boundary of the 3-ball B^3 is required to be fixed at each stage of concordance. More precisely, let $I_1, \ldots, I_n$, denote n disjoint copies of the interval I. Two n-tangles, $T_0 = (B^3, \lambda_0)$ and $T_1 = (B^3, \lambda_1)$, are

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concordant if there is a proper smooth embedding τ of $(\cup_{i=1}^n I_i) \times I$ into $B^3 \times I$, with $\tau(\cup_{i=1}^n I_i \times \epsilon) = \lambda_\epsilon$ ($\epsilon = 0, 1$) and $\tau(\epsilon_i \times I) = \tau(\epsilon_i \times 0) \times I$ for each $i = 1, \dots, n$, and $\epsilon_i = 0, 1$ in I_i. Let $(V; T_1, T_2)$ denote $(B^3 \times I, \tau)$, the concordance between T_1 and T_2. If $(V; T_1, T_2)$ and $(V'; T_2, T_3)$ are two concordances, we can then paste V' to V along T_2 to get a concordance $(V \cup V'; T_1, T_3)$. A concordance $(V; T_1, T_2)$ is *invertible at* T_2 if there is a concordance $(V'; T_2, T_1)$ such that $(V \cup V'; T_1, T_1)$ is diffeomorphic to $(T_1 \times I; T_1, T_1)$ by a diffeomorphism φ with $\varphi(\tau) = \lambda_1 \times I$, where τ is the embedding of n disjoint copies of $I \times I$ into $B^3 \times I$ defining the concordance $(V \cup V'; T_1, T_1)$ and λ_1 is the embedding of n disjoint copies of I into B^3 defining the tangle T_1.

A knot is called *doubly null concordant* if it is the slice of some unknotted 2-sphere in S^4. Two knots K_1 and K_2 are said to be *doubly concordant* if $K_1 \# J_1$ is isotopic to $K_2 \# J_2$ for some doubly null concordant knots J_1 and J_2.

The following theorem is due to Zeeman.

Theorem 2.1. [12] *Every 1-twist-spun knot is unknotted.*

Let $-K$ denote the knot obtained by taking the image of K, with reversed orientation, under a reflection of S^3. The following fact was first proved by Stallings and now follows readily from 2.1. (One cross-section of the 1-twist-spin of K yields $K \# (-K)$. For details, see [11].)

Corollary 2.2. *$K \# (-K)$ is doubly null concordant for every knot K.*

Corollary 2.3. *If $K_1 \# (-K_2)$ is doubly null concordant then K_1 and K_2 are doubly concordant.*

Proof. Take $J_1 = K_2 \# (-K_2)$ and $J_2 = K_1 \# (-K_2)$ in the definition of double concordance. □

Remark 2.4. An easy exercise shows that knots K_1 and K_2 are concordant if and only if $K_1 \# (-K_2)$ is *slice*, *i.e.*, concordant to the unknot. This defines an equivalence relation. However, a definition of double concordance more along the lines of concordance is as of yet inaccessible. The difficulty is that it is unknown whether the following is true: If knots K and $K \# J$ are doubly null concordant, then J is doubly null concordant.

There is a relation between invertible concordance and double concordance.

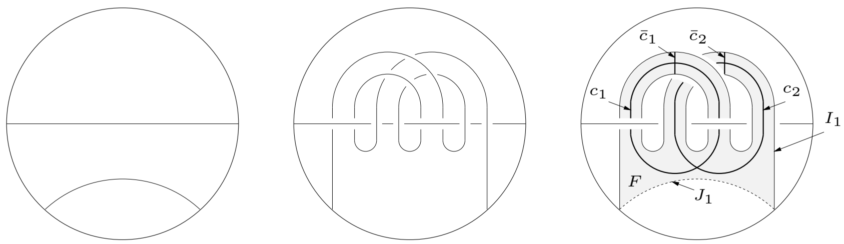
Proposition 2.5. *If K_1 is invertibly concordant to K_2 then $K_1 \# (-K_2)$ is doubly null concordant.*

Proof. There is a copy of $S^3 \times I$ in S^4 intersecting the 1-twist-spin of K_1 in $K_1 \# (-K_1) \times I$. Since K_2 splits $K_1 \times I$, there is an invertible concordance from $K_1 \# (-K_1)$ to $K_1 \# (-K_2)$. Hence $K_1 \# (-K_1) \times I$ is split by $K_1 \# (-K_2)$ and the result follows. □

3. Invertible concordances and prime knots

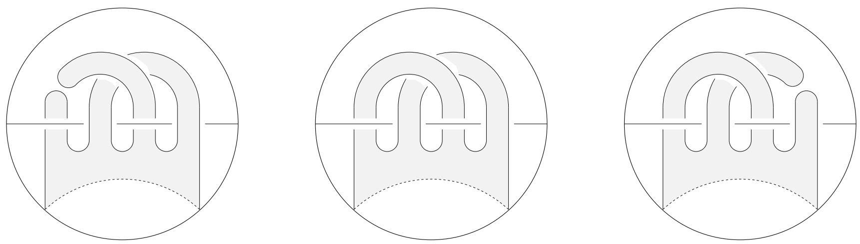
Kirby and Lickorish [1] proved that any knot in S^3 is concordant to a prime knot. Livingston [3] gave a different proof of this result using satellite knots. In this section, we modify Livingston's approach to prove Theorem 1.1.

Before proving this, we will set up some notation. By a *splitting*-S^2, S, for a knot K (in S^3 or $S^1 \times D^2$) we denote an embedded 2-sphere, S, intersecting K in



(a) U_2 (b) T (c)

FIGURE 1.



$B^3 \times \frac{1}{4}$ $B^3 \times \frac{1}{2}$ $B^3 \times \frac{3}{4}$

FIGURE 2.

exactly 2 points. A knot in either S^3 or $S^1 \times D^2$ is *prime* if for every splitting-S^2, S, S bounds some 3-ball, B, with $(B, B \cap K)$ a trivial pair. The *winding number* of a knot K in $S^1 \times D^2$ is that element z of $\mathbb{Z} \cong H_1(S^1 \times D^2; \mathbb{Z})$ with $z \geq 0$ and K representing z. The *wrapping number* of K is the minimum number of intersections of K with a disk D in $S^1 \times D^2$ with $\partial D =$ meridian. If K_1 is a knot in $S^1 \times D^2$ and K_2 is a knot in S^3, the K_1 *satellite of* K_2 is the knot in S^3 formed by mapping $S^1 \times D^2$ into the regular neighborhood of K_2, $N(K_2)$, and considering the image of K_1 under this map. The only restriction on the map of $S^1 \times D^2$ into $N(K_2)$ is that it maps a meridian to a meridian. In what follows we will consider $S^1 \times D^2$ embedded in S^3 in a standard way. Hence any knot K in $S^1 \times D^2$ gives rise to a knot K^* in S^3.

The following theorem is due to Livingston.

Theorem 3.1. [3] *Let K_1 be a knot in $S^1 \times D^2$ such that K_1^* is the unknot in S^3. Then K_1 is prime in $S^1 \times D^2$. Moreover, if K_1 has wrapping number > 1 and K_2 is any nontrivial knot in S^3, then the K_1 satellite of K_2 is prime in S^3.*

This theorem suggests that, to prove our main theorem 1.1, we only need to find a knot K_1 in $S^1 \times D^2$ with K_1^* the unknot in S^3 and an invertible concordance between the core C and the knot K_1 in $S^1 \times D^2$. To do this, we observe that there is an invertible concordance between the tangles U_2 and T in Figure 1. We remark here that Ruberman in [7] has used the tangle T to prove that any closed orientable 3-manifold is invertibly homology cobordant to a hyperbolic 3-manifold.

Lemma 3.2. *The 2-tangle T in Figure 1(b) splits $U_2 \times I$.*

Proof. Let I_1 be a copy of the non-straight arc of T in the 3-ball B^3 and let J_1 be a copy of the non-straight arc of U_2 in B^3 as shown in Figure 1(c). The closed

curve $J_1 \cup I_1$ bounds an obvious punctured torus F that is the shaded region in Figure 1(c). Consider F as the plumbing of two $S^1 \times I$. Let $c_i,\ i = 1, 2$, be the cores of the two $S^1 \times I$ of F and let $\bar{c}_i,\ i = 1, 2$, be disjoint proper line segments in F intersecting with c_i exactly once, respectively. See Figure 1(c).

To construct an invertible concordance, we will construct two concordances and then paste them together. First, note that pinching I_1 along $\bar{c}_1$ transforms T into the tangle U_2 with an unlinked unknotted circle inside which is isotopic to the circle c_2. Now capping off this circle we have a concordance $(V_1'; T, U_2)$. The tangle $B^3 \times \frac{1}{4}$ in Figure 2 represents a slice of this concordance before capping off the circle. In the similar way, pinching I_1 along $\bar{c}_2$ and capping off the unknot gives us another concordance $(V_2; T, U_2)$. Let $(V_1; U_2, T)$ denote the concordance $(V_1'; T, U_2)$ with reversed orientation. We can then paste V_1 to V_2 along T to get a concordance $(V_1 \cup V_2; U_2, U_2)$, which will be proved to be isotopic to the product concordance $U_2 \times I$. A few cross-sections of concordance $V_1 \cup V_2$ are drawn in Figure 2.

Let τ denote the embedding of two disjoint copies of $I \times I$ into $V_1 \cup V_2$ as in the definition of concordance in Section 2. It is obvious from Figure 2 that there is a 3-manifold M (the union of shaded regions) in $V_1 \cup V_2$ bounded by τ and $J_1 \times I$, whose intersection with U_2 at each end of the concordance is the arc J_1 and whose cross-section in the middle is the punctured torus F. This 3-manifold M can be considered as the union of three submanifolds: the product $F \times I$ and two 3-dimensional 2-handles $D^2 \times I$. One $D^2 \times I$ is glued to $F \times I$ along a regular neighborhood of c_2, which corresponds to capping off the circle isotopic to c_2 as we constructed the concordance V_1'. The other $D^2 \times I$ is glued along a regular neighborhood of c_1, which corresponds to capping off the circle isotopic to c_1 as we constructed the concordance V_2. Since $F \times I$ is a 3-dimensional handlebody with 2 handles with cores c_1 and c_2, M is the manifold that results by adding two 2-handles to a genus 2 solid handlebody along the cores of the 1-handles, in this case yielding B^3. Moreover, M does not intersect the other straight arc of T at any stage. Using this 3-ball M, we can isotop τ to $J_1 \times I$ in a regular neighborhood of M not disturbing the other arc and ∂B^3. This completes the proof. □

Proposition 3.3. *The knot K_1 in Figure 3(b) splits $C \times I$, where C is the core in $S^1 \times D^2$.*

Proof. Consider $S^1 \times D^2$ as the complement of the unknot m in S^3. The knot K_1 in Figure 3(b) is isotopic to K_1 in Figure 3(a). It is obvious from Figure 3(a) that $K_1 \cup m$ is the link in S^3 formed by replacing a trivial 2-tangle in Hopf link with T (dotted circle in Figure 3(a)). The proposition now follows from Lemma 3.2. □

Now we are ready to prove our main theorem 1.1.

Proof of Theorem 1.1. Let K be a knot in S^3. If K is trivial it is prime itself. Suppose now that K is nontrivial. Let K' be K_1 satellite of K where K_1 is the knot in $S^1 \times D^2$ in Figure 3(b). By Proposition 3.3, K' splits $K \times I$. We now only need to show K' is prime. Since K_1^* is the unknot in S^3, K_1 is prime by Theorem 3.1 and to complete proof it remains to show its wrapping number > 1. Its winding number is 1, hence its wrapping number is at least one. It is easy to see that the only prime knot in $S^1 \times D^2$ with wrapping number 1 is the core. So, if K_1 had wrapping number 1, then it is isotopic to the core of $S^1 \times D^2$. The -1 surgery on the meridian curve m in S^3 should make K_1^* unchanged, *i.e.*, unknotted.

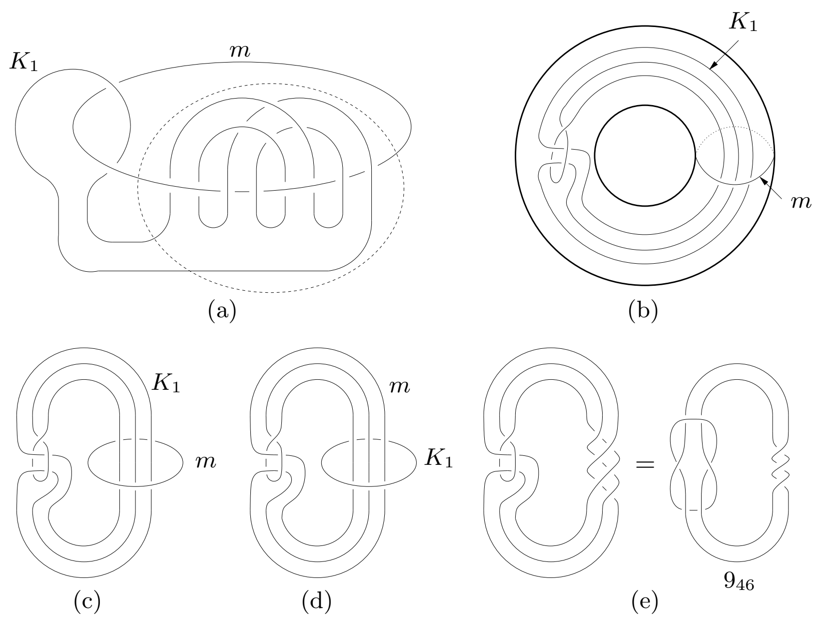


FIGURE 3.

However, the knot in Figure 3(e), the result of K_1^* after -1 surgery along m, is 9_{46} and hence knotted. Therefore the wrapping number is > 1. $\square$

Corollary 3.4. *Any knot is doubly concordant to a prime knot.*

Remark 3.5. The K_1 satellite of K has the same Alexander polynomial as that of K. Seifert [8] proved that the Alexander polynomial of the K_1 satellite of K is $\Delta_{K_1^*}(t)\Delta_K(t^w)$ if w is the winding number of K_1 in $S^1 \times D^2$. In our case, w is 1 and K_1^* is the unknot.

In [3], Livingston also proved that every 3-manifold is homology cobordant to an irreducible 3-manifold. Two 3-manifolds, M_1 and M_2, are *homology cobordant* if there is a 4-manifold W, with $\partial W = M_1 \cup M_2$ and the map of $H_*(M_i;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ an isomorphism. Invertible homology cobordisms can be defined in the same way as in the knot concordance case. A 3-manifold M is *irreducible* if every embedded S^2 in M bounds an embedded B^3.

Remark 3.6. In spirit of [3], we have a simple proof that every 3-manifold is invertibly homology cobordant to an irreducible 3-manifold. To prove this, we only need to slightly modify the proof of Theorem 3.2 in [3] by using K_1 in Figure 3(b). The -1 surgery on K_1 makes the meridian m the knot 9_{46}.

This remark is also a corollary of Ruberman's Theorem 2.6 in [7] that reads: for every closed orientable 3-manifold N, there is a hyperbolic 3-manifold M, and an invertible homology cobordism from M to N. The remark follows since a hyperbolic 3-manifold is irreducible.

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